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# Special function related to the concave–convex boundary problem of the diffraction theory

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**Abstract**

The concave–convex boundary problem of the diffraction theory is studied. It corresponds to the scattering of a whispering gallery mode on the point of inflection of the boundary. A new special function related to this boundary problem is introduced and its particular properties are discussed. This special function is defined as a contour integral on the complex plane and its behaviour in different domains of parameters is considered.

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## 1. Introduction

Whispering gallery modes are well-known objects of the diffraction theory. These modes are high-frequency waves localized in the vicinity of the boundary. The corresponding asymptotic theory can be found in [1, 2]. The results have been applied in different fields of physics, particularly, in quantum optics [3, 4]. The main existence condition for these modes is the positivity of the effective boundary curvature [1]. Hence, a question arises: how do whispering gallery modes scatter on those points of the boundary where this condition fails. The isolated point of the boundary where the effective curvature changes its sign is called the point of inflection of the boundary [5–12]. If a non-negative effective curvature takes zero value in an isolated point of the boundary, this point is called the point of local straightening of the boundary [13–16]. We call both types of points singular points of the boundary.

The scattering of the whispering gallery modes on points of inflection and on straightening points is a challenging problem of the diffraction theory. In the case of an inflection point it is called the concave–convex boundary problem. The problems formulated above were under consideration for about three decades, but no complete analytical solution has been obtained. Even the description of the wave field in a neighbourhood of the singular point is lacking. The first step in this direction might be the construction of the corresponding special functions similar to the Airy function but more sophisticated.

Note, that special functions related to wave catastrophes were discussed in [17–19]. Namely, the following integrals were considered in these papers

$$u(\alpha) = \int_{-\infty}^{+\infty} \exp[\mathrm{i}\omega\Gamma(\rho, \alpha)]\,\mathrm{d}\rho$$

where the integration variable $\rho$ and the variable $\alpha$ take their values in some vector spaces and the function $\Gamma(\rho, \alpha)$ is a polynomial. In this paper, we discuss the special function related to the above-mentioned problem of the diffraction theory and this special function is a sort of realization of the programme outlined in [17–19] for this concrete problem.

Let us sketch the plan of the paper. Firstly, we recall some facts about whispering gallery modes. Then we describe a variant of a 'separation of variables', which generates a rich set of solutions for the model partial differential equation. In section 4, we describe the special function related to the concave–convex problem of the diffraction theory. This function is a contour integral on the complex plane. This integral is characterized by a large parameter $\omega \gg 1$ which is the wave frequency. We consider the main properties of this integral including the location of saddle points, their movements under variations of parameters, the location and the character of critical lines and points.

## 2. Whispering gallery modes

### *2.1. Physical formulation of the problem*

The whispering gallery modes in two-dimensional geometries can be described in the parabolic equation framework [1]. We are interested in the solutions of the following boundary problem:

$$\left[\Delta + \frac{\omega^2}{c^2(x, y)}\right] u(x, y) = 0 \qquad x, y \in \Omega \tag{1}$$

$$u(x, y)|_{\partial\Omega} = 0 \tag{2}$$

localized asymptotically near the smooth boundary $\Omega$, $\omega \gg 1$. Here $c(x, y)$ is the velocity of the wave propagation, $\omega$ is the frequency of the oscillations. Let us introduce semigeodesic coordinates $(s, n)$ in a neighbourhood of the boundary, with $s$ being the length of the geodesic line along the boundary, $n$ being the length of the perpendicular from the point $M = (s, n)$ to the boundary and $n < 0$ for $M \in \Omega$. Substituting

$$u(s, n) = U(s, n) \exp\left[\mathrm{i}\omega \int_{s_0}^{s} \frac{\mathrm{d}s}{c(s, 0)}\right]$$

for a fixed $s_0$ and rewriting equation (1) in the new coordinates, we obtain

$$\left(1 + \frac{n}{\rho(s)}\right) U_{nn} + \left(1 + \frac{n}{\rho(s)}\right)^{-1} U_{ss} + \left\{\frac{2\mathrm{i}\omega}{c_0(s)} \left(1 + \frac{n}{\rho(s)}\right)^{-1} + \frac{\partial}{\partial s}\left(\frac{1}{1 + \frac{n}{\rho(s)}}\right)\right\} U_s$$
$$+ \left\{\frac{1}{\rho(s)} U_n + \mathrm{i}\omega \frac{\partial}{\partial s}\left(\frac{1}{c_0(s)\left(1 + \frac{n}{\rho(s)}\right)}\right) + \left(1 + \frac{n}{\rho(s)}\right) \frac{\omega^2}{c^2(s, n)}\right.$$
$$\left. - \left(1 + \frac{n}{\rho(s)}\right)^{-1} \frac{\omega^2}{c_0^2(s)}\right\} U = 0.$$

Here $\rho(s)$ is the radius of the boundary curvature, $c_0(s) \equiv c(s, 0)$. From physical considerations it follows that we have to adopt $n = O(\omega^{-2/3})$, $U = O(1)$, $U_s = O(\omega^{1/3})$,

$U_n = O(\omega^{2/3}), U_{ss} = O(\omega^{2/3}), U_{nn} = O(\omega^{4/3})$ for the whispering gallery mode. Keeping the terms $O(\omega^{4/3})$, $O(\omega)$, we arrive at the parabolic equation

$$U_{nn} + \frac{2\mathrm{i}\omega}{c_0(s)}U_s + \left\{\mathrm{i}\omega\frac{\partial}{\partial s}\left(\frac{1}{c_0(s)}\right) + \omega^2\frac{2n}{c_0^2(s)P(s)}\right\}U = 0 \tag{3}$$

where $P(s)$ is an 'effective' radius of curvature,

$$\frac{1}{P(s)} \equiv \frac{1}{\rho(s)} - \frac{c_1(s)}{c_0(s)}$$

and the function $c_1(s)$ is defined by the expansion

$$\frac{1}{c(s,n)} = \frac{1}{c_0(s)}\left[1 - \frac{c_1(s)}{c_0(s)}n + \cdots\right].$$

Parabolic equation (3) is the basis of the following considerations. If $P(s) > 0$ for any $s$, the corresponding asymptotic procedure generates the asymptotic expansion of a solution of the boundary problem (1), (2).

### 2.2. *Solution of the model problem*

The theory of the whispering gallery mode is based on application of the Airy function [1, 2]. Here we consider the simplest model problem in order to introduce this function. At $z \geqslant 0$, $-\infty < t < \infty$, we seek for a solution of the equation

$$\mathrm{i}\omega\Phi_t(z,t) + \Phi_{zz}(z,t) - \omega^2 z\Phi(z,t) = 0 \tag{4}$$

satisfying boundary conditions:

$$\Phi(z,t) \in L_2(0,\infty) \tag{5}$$

at any $t$, and

$$\Phi(0,t) = 0 \tag{6}$$

the Dirichlet boundary condition. As was assumed above, the frequency $\omega \gg 1$ is the large parameter of the problem. Note, that the coefficient in front of $\omega^2 z$ in the last equation is equal to $(-1)$. It corresponds to the positivity of the effective boundary curvature. Equation (5) can be obtained from (3) at $c(s,n) = 2$, $P(s) = 1$. In order to construct the asymptotics of the whispering gallery mode, i.e. the solution of the boundary problem (4), (5), (6), we reproduce the consideration of the item 5.3 of [1] (with simplifications involved by the simpler model equation).

In the leading asymptotic order we need to solve the equation

$$\Phi_{zz}(z,t) - \omega^2 z\Phi(z,t) = 0.$$

Its solution is obtained in the form of the Airy integral

$$\Phi_0(z,t) = \int_L \exp\{\mathrm{i}\omega(zs + s^3/3)\}\,\mathrm{d}s. \tag{7}$$

Here the contour of integration is chosen in such a way that the integrand decreases exponentially at $z \to +\infty$. It is the scaled Airy function, i.e. $\Phi_0(z,t) = W(\omega^{2/3}z)$, and $W(x)$ is the corresponding solution of the Airy equation

$$W_{xx}(x) - xW(x) = 0.$$

If we substitute the function $\Phi_0(z,t)$ in (4) this equation is satisfied in the leading asymptotic order. Further, in order to fit the boundary condition at $z = 0$, we suppose

$$\Phi_1(z,t) = A(t)W(\omega^{2/3}z + \gamma_p) \tag{8}$$

where $\gamma_p$ is a root of the Airy function $W(x)$. Substituting the latter function in (4), we choose function $A(t)$ in order to eliminate the next asymptotic terms. We obtain

$$\Phi_1(z,t) = \exp(\mathrm{i}\omega^{1/3}\gamma_p t)W(\omega^{2/3}z + \gamma_p). \quad (9)$$

This function is an approximate solution of the boundary problem described above. One can derive the full asymptotic expansion on its basis (the corresponding discussion can be found in [1, 2]). In the vicinity of the point $z = 0$ the Airy integral oscillates. Hence, the whispering gallery mode also oscillates in the $\omega^{-2/3}$ vicinity of the boundary, and its amplitude decreases exponentially as $z$ increases.

The more detailed analysis based on [1] leads to the conclusion that the success of the asymptotic procedure was ensured by a proper choice of the initial ansatz and the usage of the Airy function. The structure of this function reflects the main features of the wave field in the vicinity of the boundary and assists in establishing the recurrent procedure for constructing the full asymptotic expansion of the solution of the boundary problem. In some sense the Airy function is a 'load-carrying structure' for the whispering gallery mode.

**Remark 1.** When choosing the leading element of the asymptotic ansatz (the Airy function in this case) the boundary condition at $z = 0$ plays the minor role. As follows from the results of [1, 2], this condition influences only the subdominant terms of the wave field.

**Remark 2.** The Airy function is a contour integral on the complex plane with holomorphic integrand. The same holds for the special functions described below. It means, that the contour of integration is fixed by the ends of the contour. The integrands include the large parameter and for evaluation of the integrals one can use the saddle point method [20]. The asymptotic contribution of the saddle point depends on its order $k$ [20]. If the contour $L_1$ passes through the saddle point $v_s$ of order $k$, then

$$\int_{L_1} T(v)\exp(\mathrm{i}\omega P(v))\,\mathrm{d}v \sim \omega^{-1/(k+1)}$$

(if $T(v_s) \neq 0$). Therefore, with increasing order of the saddle point the wave amplitude also increases (the focusing of the wave field). This fact is well known for the Airy function, and it is associated with the coalescence of two simple saddle points into one double saddle point in the vicinity of the caustic [1, 2].

## 3. 'Separation' of variables

### *3.1. General case*

In this section, we present the method which generates the rich set of solutions of some partial differential equations. We start from the general construction and then consider the equations most interesting for us. For this reason we study the following partial differential equation:

$$\mathrm{i}\omega\Psi_t + \Phi_{zz} + \omega^2 f(t)z\Psi = 0. \quad (10)$$

Here $f(t)$ is an arbitrary (for the present) function, $\omega \gg 1$ is a large parameter. Let us seek the solutions of equation (10) in the following form of an integral:

$$\Psi(z,t) = \int_L \exp[\mathrm{i}\omega(zs - ts^2)]q(t,s)\,\mathrm{d}s \quad (11)$$

where $L$ is some appropriate contour of integration, which guarantees the possibility of integration by parts. Substituting (11) into the left-hand side of equation (10), we obtain

$$R = \mathrm{i}\omega\Psi_t + \Psi_{zz} + \omega^2 f(t)z\Psi = \int_L \exp[\mathrm{i}\omega(zs - ts^2)][\mathrm{i}\omega q_t + \omega^2 f(t)zq]\,\mathrm{d}s. \quad (12)$$

Note, that

$$\omega^2 z \exp(\mathrm{i}\omega z s) = -\mathrm{i}\omega \frac{\mathrm{d}}{\mathrm{d}s} \exp(\mathrm{i}\omega z s).$$

Integrating by parts in (12), we get

$$R = \int_L \mathrm{d}s \exp[\mathrm{i}\omega(-ts^2 + zs)][\mathrm{i}\omega q_t + \mathrm{i}\omega f(t)(q_s - 2\mathrm{i}\omega t s q)] = 0.$$

The last relation is fulfilled if

$$q_t + f(t)q_s = 2\mathrm{i}\omega t f(t) s q.$$

This partial differential equation is of the first order and can be solved by separation of variables. Let

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = -f(t). \tag{13}$$

After substitution

$$\tau = t \qquad y = s + \lambda(t)$$

we arrive with new variables at the simple equation

$$q_\tau = 2\mathrm{i}\omega\tau f(\tau)(y - \lambda(\tau))q.$$

This equation can be solved and its solution reads

$$q(t,s) = \exp\left[2\mathrm{i}\omega \int_0^t \xi f(\xi)(s + \lambda(t) - \lambda(\xi))\,\mathrm{d}\xi\right] Q(s + \lambda(t))$$

where $Q(\zeta)$ is an arbitrary function. Returning to the initial variables, we obtain

$$\Psi(z,t) = \int_L \exp\left[\mathrm{i}\omega\left(zs - ts^2 + 2\int_0^t \xi f(\xi)\,(s + \lambda(t) - \lambda(\xi))\,\mathrm{d}\xi\right)\right] Q(s + \lambda(t))\,\mathrm{d}s.$$

This relation expresses the solutions of equation (10). It contains two undetermined objects—the function $Q(\zeta)$ and the contour of integration $L$. We will define the contour later, comparing the behaviour of $\Psi(z,t)$ at $t \to -\infty$ with known asymptotics ([5], see below). Here we consider the choice of function $Q(\zeta)$.

Let us suppose that

$$Q(\zeta) = \exp[\mathrm{i}\omega Q_0(\zeta)]T(\zeta)$$

where the function $T(\zeta)$ is related to the subdominant asymptotic terms. Note, that

$$\Psi(0,t) = \int_L \exp\left[\mathrm{i}\omega\left(-ts^2 + Q_0(s + \lambda(t)) + 2\int_0^t \xi f(\xi)\,(s + \lambda(t) - \lambda(\xi))\,\mathrm{d}\xi\right)\right] T(s + \lambda(t))\,\mathrm{d}s. \tag{14}$$

The function $Q_0(\zeta)$ is fixed by the following condition A.

**Condition A.** *One saddle point of the integrand in (14) coincides with the point* $s = 0$.

The reasons for this condition are the following. The integrand in (14) has several saddle points depending on the value of $t$ for arbitrary function $Q_0(\zeta)$. Using the shift of the integration variable we can fix one of these saddle points at the point $s = 0$. Therefore, condition A is ‘natural’ and does not restrict the generality of our approach. Note, that this condition is fulfilled for the Airy integral.

The following relation is a consequence of condition A:

$$Q_0'(\lambda(t)) + 2\int_0^t \xi f(\xi)\,\mathrm{d}\xi = 0.$$

Differentiation with respect to $t$ leads to

$$Q_0''(\lambda(t))\lambda'(t) + 2tf(t) = 0.$$

Taking into account relation (13), the last formula can be rewritten as

$$Q_0''(\lambda(t)) = 2t. \tag{15}$$

For a given function $f(t)$ relations (13), (15) can be considered as a system of functional-differential equations for functions $Q_0(\zeta)$, $\lambda(t)$, which can be solved in some interesting cases. As a result, we obtain for the solutions of partial differential equation (10) the following integral:

$$\begin{aligned}\Psi(z,t) &= \int_L \mathrm{d}s\,\exp\{\mathrm{i}\omega M(z,t,s)\}T(s+\lambda(t))\\ M(z,t,s) &= \left[zs - ts^2 + Q_0(s+\lambda(t) + 2\int_0^t \xi f(\xi)(s+\lambda(t)-\lambda(\xi))\,\mathrm{d}\xi\right].\end{aligned} \tag{16}$$

Here we suppose that $T(s)$ is an analytic function and the integral is convergent.

### 3.2. *Boundary with constant curvature*

Firstly, we consider the case $f(t) = -1$, which corresponds to the boundary with constant curvature. It follows from equations (13), (15): $\lambda(t) = t$, $Q_0(t) = t^3/3$. Substituting these relations into (16) we get ($T(x) = 1$):

$$M(z,t,s) = zs + s^3/3.$$

Hence, we reproduce in this situation the well-known Airy integral (7).

### 3.3. *Concave–convex boundary*

Here we study the case $f(t) = 2t$, taking into account the form of the corresponding model equation (see equation (19) below). In this case equations (13) and (15) can be solved in explicit form. We obtain

$$\lambda(t) = -t^2 \qquad t = (-\lambda)^{1/2}.$$

Then

$$Q_0(\lambda) = \tfrac{8}{15}(-\lambda)^{5/2}$$

and

$$\begin{aligned}\Psi(z,t) &= \int_L \exp[\mathrm{i}\omega M(z,t,s)]T(s+\lambda(t))\,\mathrm{d}s\\ M(z,t,s) &= zs - ts^2 + \frac{4st^3}{3} - \frac{8t^5}{15} + \frac{8}{15}(t^2-s)^{5/2}.\end{aligned} \tag{17}$$

One can rewrite these relations removing the turning points in integrand (17). After substitution

$$t^2 - s = v^2$$

we find

$$\Psi(z,t) = \int_L \exp[\mathrm{i}\omega N(t,z,v)]T(v)\,\mathrm{d}v \tag{18}$$

where

$$N(t,z,v) = z(t^2-v^2) - t(t^2-v^2)^2 - \frac{8t^5}{15} + \frac{4t^3}{3}(t^2-v^2) + \frac{8}{15}v^5$$

and $T(v)$ is an arbitrary analytic function with the corresponding behaviour at infinity.

Let us discuss the possible choice of integration contour $L$ in (18). We suppose that $\omega > 0$ implying that the ends of contour must be in sectors $(2\pi n/5, \pi(2n+1)/5)$, $n = 0, 1, 2, 3, 4$. If the function $T(v)$ grows at infinity not too fast, our choice of contours provides the convergence of integrals and the possibility of integration by parts.

## 4. Special function for the concave–convex boundary

We consider in this section the special function which is an analogue of the Airy function for the concave–convex boundary problem of diffraction theory. Namely, we study the following model problem: to find a solution of the equation

$$\mathrm{i}\omega\Psi_t + \Psi_{zz} + 2\omega^2 tz\Psi = 0 \tag{19}$$

which satisfies the boundary condition (5) and reproduces at $t \to -\infty$ the scaled Airy function or, in other words, the incoming whispering gallery mode (see [5, 6, 9]). As mentioned above, the boundary condition at $z = 0$ determines only the subdominant terms of the asymptotics.

Note, that after scaling

$$t = 2^{-3/5}\omega^{-1/5}\tau \qquad z = 2^{1/5}\omega^{-3/5}\zeta \tag{20}$$

equation (19) can be rewritten without parameter $\omega$ as

$$\mathrm{i}\Psi_\tau + \tfrac{1}{2}\Psi_{\zeta\zeta} + \tau\zeta\Psi = 0 \tag{21}$$

(see [6, 7, 9, 14]). It means that the asymptotic considerations can be used outside the vicinity of the point $(z,t) = (0,0) : t = O(\omega^{-1/5})$, $z = O(\omega^{-3/5})$.

Now we study the special function defined by the contour integral

$$\Psi(z,t) = \int_L \exp[\mathrm{i}\omega N(t,z,v)]\,\mathrm{d}v \tag{22}$$

where

$$N(t,z,v) = z(t^2-v^2) - t(t^2-v^2)^2 - \frac{8t^5}{15} + \frac{4t^3}{3}(t^2-v^2) + \frac{8}{15}v^5.$$

Here the contour $L$ begins in the sector $(4\pi/5, \pi)$ at infinity, and ends in the sector $(2\pi/5, 3\pi/5)$ at infinity. We call this function the *concave–convex special function* (CCSF).

The more general expression

$$\Psi(z,t) = \int_L \exp[\mathrm{i}\omega N(t,z,v)]T(v)\,\mathrm{d}v$$

can also be studied. Here $T(v)$ is an arbitrary analytical function with the corresponding behaviour at infinity. These integrals might be useful while examining the full asymptotic expansion of our model problem.

As shown above, special function (22) is a solution of equation (19) at any $z, t$. We consider here the set of saddle points of this integral.

### *4.1. Saddle points*

In the general case integral (22) has four simple saddle points. The equation for these points is

$$\frac{\mathrm{d}N(t,z,v)}{\mathrm{d}v} = -2zv - 4tv^3 + \frac{4t^3v}{3} + \frac{8v^4}{3} = 0. \tag{23}$$

If a double saddle point arises, we have an additional equation

$$\frac{\mathrm{d}^2N(t,z,v)}{\mathrm{d}v^2} = -2z + \frac{4t^3v}{3} - 12tv^2 + \frac{32v^3}{3} = 0. \tag{24}$$

After some algebraic manipulations with relations (23) and (24) one can obtain

$$(t - v)v^3 = 0.$$

Hence, if the order of the saddle point is larger than unity there are two possibilities.

A. For

$$v = 0 \qquad z = \frac{2t^3}{3} \tag{25}$$

we have a saddle point of the third order as a result of $\mathrm{d}^3N(t,z,v)/\mathrm{d}v^3 = 0$. The first derivative which is not equal to zero at this point is of the fourth order.

B. For

$$v = t \qquad z = 0$$

the third derivative with respect to $v$ is not equal to zero with the result that the saddle point is of second order. But this point gives the contribution in our integral for $t < 0$ only. It means that the line $z = 0$ is critical for our integral for $t < 0$ only.

Note, that at the point $z = t = 0$ all four saddle simple points coalesce. In this case the saddle point is of fourth order.

As mentioned in remark 2, the increase in order of a saddle point leads to a corresponding increase in the asymptotic contribution of this saddle point, or, in other words, to an increase in the intensity of the wave field (the wave field is 'focusing'). The line $z = 2t^3/3$ corresponds to the tangent line for the boundary passing through the point of inflection, see [7, 9]). So, this critical line for CCSF arises as a line of multiple saddle points. In Russian mathematical literature it is called the 'searchlight' line.

Let us rewrite the polynomial $N(t, z, v)$, taking into account relation (25) and supposing

$$z = 2t^3/3 + \eta.$$

After this substitution

$$N(t,z,v) = R(t,\eta,v) + r(\eta,t)$$
$$R(t,\eta,v) = \frac{8v^5}{15} - tv^4 - \eta v^2$$
$$r(\eta,t) = \eta t^2 + \frac{7t^5}{15}.$$

Note, that the function $r(\eta, t)$ is a phase shift and does not influence the position of saddle points at different $t, \eta$. The polynomial $R(t, \eta, v)$ can be considered as the swallow tail singularity (specified in terms $t, \eta$, see [21, 22]).

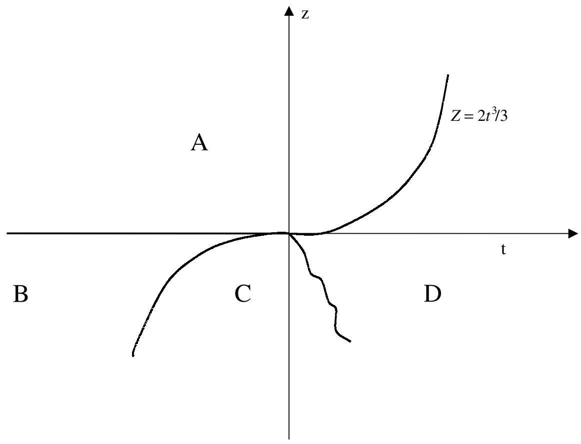


**Figure 1.** Domains with different topology of level lines on $(t, z)$-plane.

### *4.2. Transformation of saddle points*

Further study of CCSF (at $\omega \gg 1$) is based on the behaviour of the polynomial $N(t, z, v)$ and the saddle points in different domains of variables. We consider here not only the half-plane $z \geqslant 0, -\infty < t < \infty$, but the negative values of variable $z$ too.

In accordance with the general ideology of the saddle point method [20], we need to describe the behaviour of saddle points and the topology of level lines under variations of variables $z, t$. This problem is too cumbersome from the analytical point of view and we prefer to use here the computer simulations. In this section, we present the results of these computer calculations. But some preliminary information can be obtained from equation (23). It follows from this equation that all saddle points are real for real values $z, t$ in the domain between the lines $z = 0, z = 2t^3/3$. If $z, t$ is outside this domain, there are a couple of real saddle points (one of which is the point $v_0 = 0$) and a couple of complex conjugate saddle points.

1. As mentioned above, there are four simple saddle points in the general case. The point $v = 0$ is a saddle point at any $z, t$. The line $C_1 = (z, t : z = 0)$ contains points for which the corresponding saddle points (contributory for our integral) are of second order. The line $C_2 = (z, t : z = 2t^3/3)$ corresponds to the situation when the coalescence of two or three saddle points takes place. One can split the parameter plane $z, t$ into four parts in accordance with various sets of saddle points contributory to our integral (see figure 1).
2. When $t$ is negative and large enough in modulo and $z$ is positive, there is a couple of saddle points $v_+(z, t)$, $v_-(z, t)$ with identical asymptotical behaviour, $v_\pm(z, t) \sim t$, $\operatorname{Im} v_+(z, t) > 0$, $\operatorname{Im} v_-(z, t) < 0$. The point $v_+(z, t)$ is contributory for our integral (see figure 2). The same situation (the existence of one saddle point contributory for our integral) is valid in the domain $z > 2t^3/3$ and in the domain A of our figure 1. In this case $\operatorname{Re}(\mathrm{i}N(t, z, v_+)) < 0$ and CCSF decreases exponentially when $z$ tends to positive infinity. There are a couple of other saddle points, one of which is $v_0 = 0$ and the last one $v_4(z, t)$ is located on the positive part of the real axis.
3. When $t$ is negative and large enough in modulo and $z \searrow +0$, the points $v_-$ and $v_+$ coalesce and become real when $z$ takes a negative value. In this situation both these points are contributory for our integral (see figure 3). These values $z, t$ form the domain B

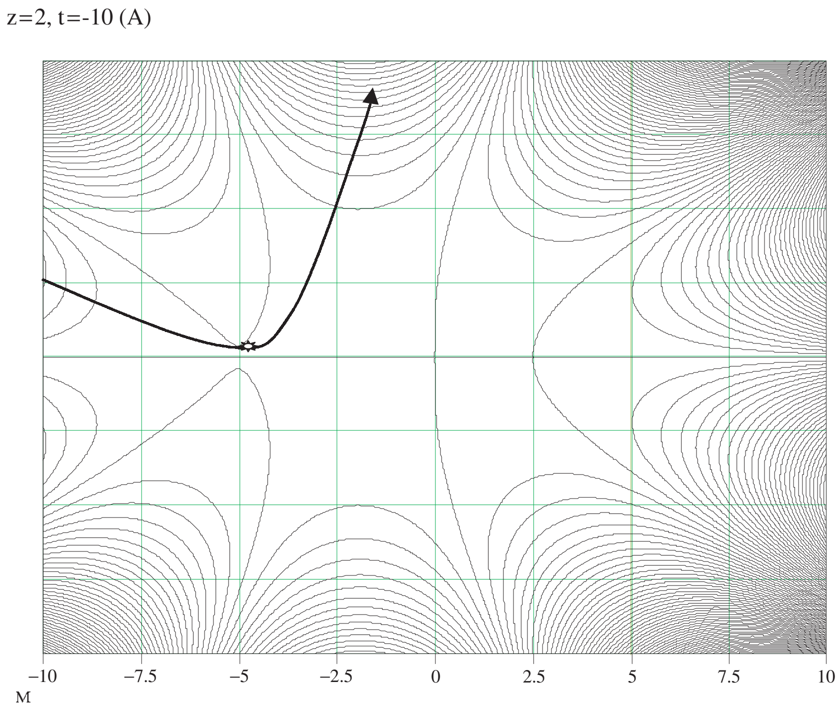


**Figure 2.** Typical level lines and contour of integration for domain A, $z = 2$, $t = -10$.

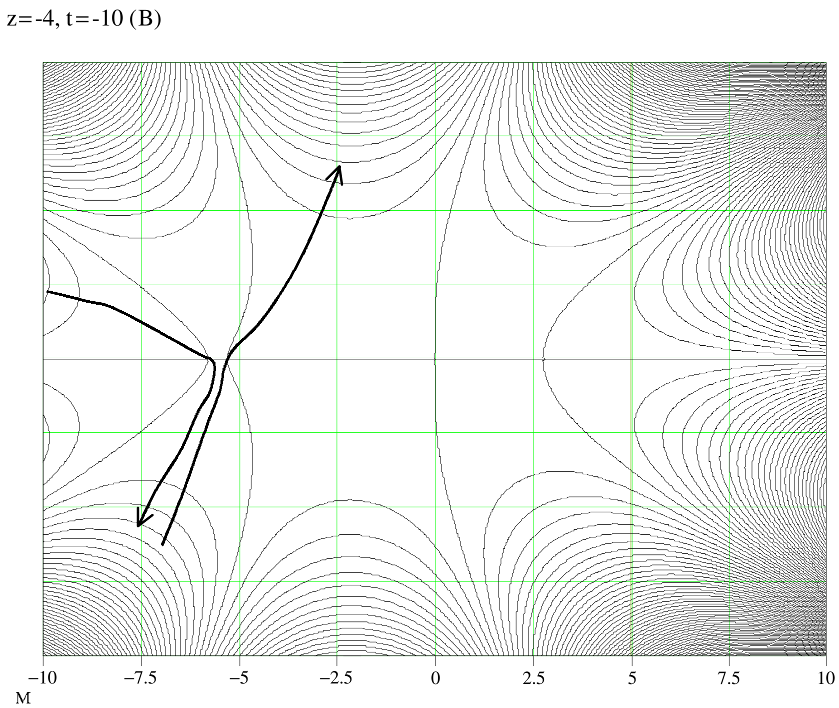


**Figure 3.** Typical level lines and contour of integration for domain B, $z = -4$, $t = -10$.

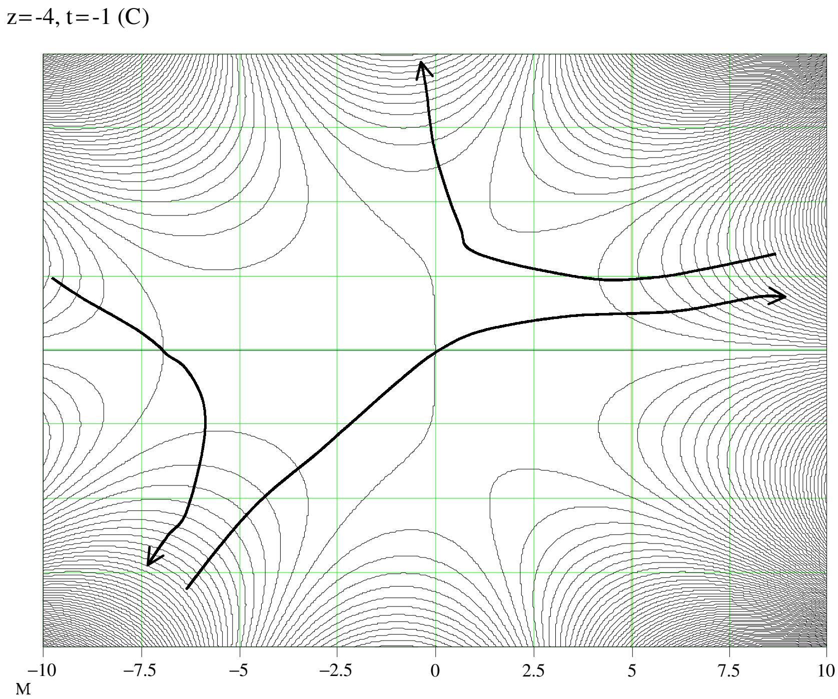


**Figure 4.** Typical level lines and contour of integration for domain C, $z = -4$, $t = -1$.

in figure 1. Saddle points $v_0$, $v_4$ (which take real values) are not contributory. Note, that $\mathrm{Re}(\mathrm{i}N(t, z, v_\pm)) = 0$.

4. When we are close to the line $z = 2t^3/3$ from the left at negative $z$, one of the points $v_+$, $v_-$ (namely, their analytic continuations; we suppose that $v_+ < v_-$, thus meaning the point $v_-$), and the points $v_0$, $v_4$ coalesce and generate a saddle point of third order. Further, when the parameter $t$ increases, the saddle points $v_-$, $v_4$ take complex values and are located to the right of the saddle point $v_0$. There is a domain on the $z, t$-plane for which we have three saddle points contributory for our integral: $v_+$, $v_0$ and the saddle point $v_c$ which is located in the upper half-plane to the right of the point $v_0$ (see figure 4). Note, that $\mathrm{Re}(\mathrm{i}N(t, z, v_{\tilde{n}})) < 0$ and the contribution of this point is exponentially small ($\mathrm{Re}(\mathrm{i}N(t, z, v_0)) = \mathrm{Re}(\mathrm{i}N(t, z, v_+)) = 0$ (similar situations were discussed in [23]). These values $z, t$ form the domain C in figure 1.
5. When $t$ increases we get the domain D in figure 1, where two saddle points $v_+$ and $v_0$ are contributory for our integral, and $\mathrm{Re}(\mathrm{i}N(t, z, v_+)) = \mathrm{Re}(\mathrm{i}N(t, z, v_0)) = 0$ (see figure 5).

### *4.3. Vicinities of critical lines and points*

Let us consider the behaviour of CCSF in the vicinities of critical points when the order of the saddle points is altering.

1. First we consider the line $z = 0$, $t < 0$. As mentioned above, at this line the saddle point of our integral $v = t$ is a double one. Substituting $v = t + s\omega^{-1/3}$, $z = \zeta\omega^{-2/3}$, we obtain

$$\omega N(z, t, v) = \zeta s(-2t) + 4t^2 s^3/3 + O(\omega^{-1/3}).$$

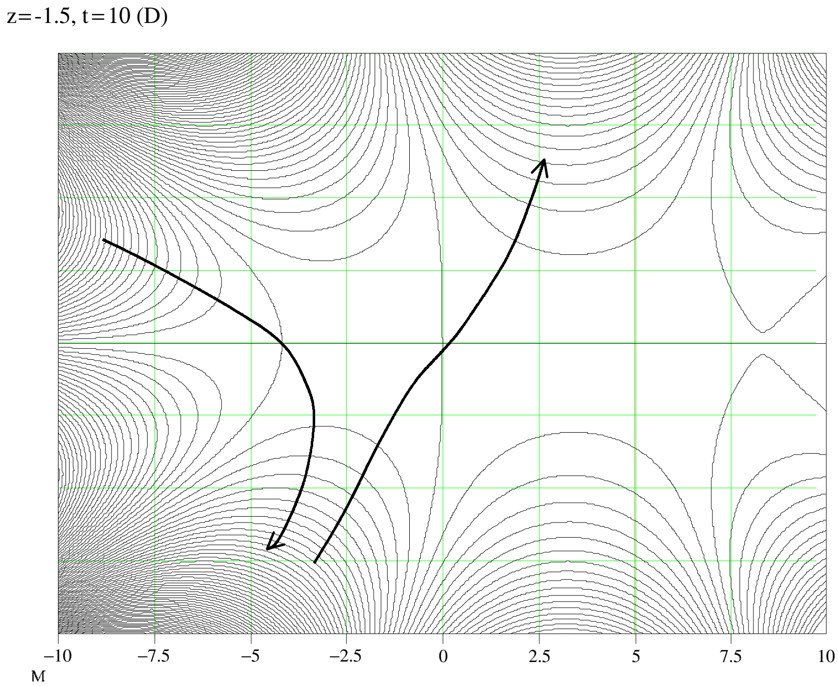


**Figure 5.** Typical level lines and contour of integration for domain D, $z = -1.5$, $t = 10$.

Hence, in the leading asymptotic order with respect to the large parameter $\omega$ we have

$$\Psi(\zeta\omega^{-2/3}, t) = \omega^{-1/3} R_1(\zeta, t)$$

where

$$R_1(\zeta, t) = \int_{L_1} \exp(\mathrm{i}(\zeta s(-2t) + 4t^2 s^3/3))\,\mathrm{d}s$$

is a scaled Airy integral (7) (the contour $L_1$ corresponds here to the contour for integral (7)). The factor $\omega^{-1/3}$ agrees with the double saddle point for the line $z = 0$. The function $R_1(\zeta, t)$ represents CCSF in the $\omega^{-2/3}$ vicinity of the critical line in the leading asymptotic order with respect to the large parameter $\omega$.

2. At the critical line $z = 2t^3/3$ our integral has the saddle point of third order. Let $\eta = z - 2t^3/3$, then

$$N(z, t, v) = 8v^5/15 - tv^4 - \eta v^2 + r(z, t)$$

where the function $r(z, t) = zt^2 - t^5/5$ does not depend on the variable of integration. After scaling

$$\eta = \omega^{-1/2}\rho \qquad v = \omega^{-1/4}s$$

we get in the leading asymptotic order in $\omega$:

$$\Psi(z, t) = \omega^{-1/4} \exp(\mathrm{i}\omega r(z, t)) R_2(\rho, t)$$

where

$$R_2(\rho, t) = \int_{L_2} \exp(-\mathrm{i}(ts^4 - \rho s^2))\,\mathrm{d}s$$

and the contour $L_2$ begins at infinity in sector $(3\pi/4, \pi)$, and ends at infinity in sector $(\pi/4, \pi/2)$ (we suppose here that $t > 0$). This contour agrees with the contour of integration in integral (22). The factor $\omega^{-1/4}$ corresponds to the triple saddle point of our integral, when $(z, t)$ belongs to the 'searchlight' line. The function $R_2(\rho, t)$ describes the behaviour of CCSF in the $\omega^{-1/2}$ vicinity of the 'searchlight' line. As follows from relation 8.3.3 of [24], the parabolic cylinder function $D_\nu(z)$ has the following integral representation:

$$\Gamma(-\nu)D_\nu(z) = \exp(-z^2/4)\int_0^\infty \exp(-zt - t^2/2)t^{-\nu-1}\,\mathrm{d}t \qquad \mathrm{Re}\,\nu < 0. \tag{26}$$

Comparing the integral representation for the function $R_2(\rho, t)$ with relation (26), we conclude that this function can be expressed in terms of the parabolic cylinder function. We omit here this cumbersome expression.

3. Let us discuss now the vicinity of the point $z = t = 0$, where our integral has the saddle point of fourth order. After scaling

$$v = \omega^{-1/5}s \qquad t = \omega^{-1/5}\tau \qquad z = \omega^{-3/5}\zeta$$

we obtain

$$\Psi(z, t) = \omega^{-1/5}\exp(\mathrm{i}\omega r(z, t))\,\Xi_1(\zeta, \tau)$$

where

$$\Xi_1(\zeta, \tau) = \int_{C_1} \exp(\mathrm{i}(8s^5/(15) - \tau s^4 + (2\tau^3/3 - \zeta)s^2))\,\mathrm{d}s$$

and the contour $C_1$ is similar to the contour $L$. This means that it begins at infinity in the sector $(4\pi/5, \pi)$ and ends at infinity in the sector $(2\pi/5, 3\pi/5)$. The function $\Xi_1(\zeta, \tau)$ represents the behaviour of CCSF $\Psi(z, t)$ in an asymptotically small neighbourhood of the point $z = t = 0$, which is the focusing point of the wave field. Note, that the function $\Xi_1(\zeta, \tau)$ is a solution of the equation

$$\mathrm{i}\Xi_{1\tau} + \Xi_{1\zeta\zeta} + 2\mathrm{i}\tau^2\Xi_{1\zeta} = 0.$$

### 4.4. *Asymptotic behaviour of the CCSF*

As was shown above, the saddle points are solutions of a cubic equation (in the general case) and can be represented in analytic form. In each domain (A, B, C, D) we know the set of contributory saddle points. This information gives us the possibility of deriving the asymptotics of CCSF at different values of $z, t$. We will not give here the corresponding boring relations and restrict ourselves to a general discussion of the behaviour of CCSF. As follows from the results above, this function decreases exponentially in the domain A when $z$ increases. In the domains B, C, D the CCSF oscillates, its amplitude decreases as a power when values $z, t$ increase.

### 4.5. *Main properties of the CCSF*

Let us summarize the main properties of CCSF $\Psi(z, t)$.

1. It is a solution of model equation (19) at any $z, t$.
2. It decreases exponentially at $z \to +\infty$.
3. It has the focusing point $z = t = 0$ and the focusing line $z = 2t^3/3$ ('searchlight' line). Recall that the last line corresponds (in physical variables) to the tangent line for the boundary at the inflection point.

4. Let us consider our function at $t \to -\infty$ in more detail. It follows from the analysis of function $N(z, t, v)$ that the integrand in (22) has two simple contributory saddle points close to $t$. After substitution

$$v = t + s(-2t)^{-2/3} \tag{27}$$

we obtain

$$N(t, z, v) = n(t, z, s) = z(-2t)^{1/3}s + s^3/3 + O(t^{-4/3}).$$

Therefore, when $t \to -\infty$ this exponent is similar to the exponent in the Airy integral (7). Moreover, our choice of the integration contour in (22) leads after (27) to the integration contour in (7). It means, that CCSF reproduces the scaled Airy function when $t \to -\infty$. As a result, we find the known asymptotics of the sought solution of the model problem [5]. Note, that this asymptotics corresponds to the whispering gallery mode approaching an inflection point.

## 5. Conclusion

We have discussed the model boundary problem corresponding to the scattering of a whispering gallery mode on the point of inflection of the boundary. It is well known that the representation of the whispering gallery mode is based on the Airy function. We have introduced a new special function, namely, the contour integral on the complex plane depending on two variables. This function can be considered as an extension of the Airy function for our model boundary problem. The explicit form of this function gives us the possibility of studying its properties at arbitrary values of variables. It is shown that this function reproduces the main asymptotic characteristics of the solution of the model boundary problem.

A similar special function can be proposed for the boundary with a point of local straightening [25].

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